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Practice Problems: Exercise 6 – Microengineering 110

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1. A gas station has both self-service and full-service islands. In the self-service island, customers fill gas themselves. In the full-service island, an attendant fills the gas On each island, there is a single regular petrol pump with two hoses. Let X denote the number of hoses being used on the self-service island at a particular time, and let Y denote the number of hoses on the full-service island in use at that time. The joint pmf of X and Y appears in the accompanying tabulation.

$p(x, y)$		y: 0	1	2
	0	.10	.04	.02
x	1	.08	.20	.06
	2	.06	.14	.30

a. What is P(X = 1 and Y = 1)?

$P(X = 1, Y = 1) = p(1,1) = .20.$

b. Compute P(X <= 1 and Y <= 1).

$P(X \le 1 \text{ and } Y \le 1) = p(0,0) + p(0,1) + p(1,0) + p(1,1) = .42.$

c. Compute the probability of X $\neq$ 0 and Y $\neq$ = 0.

$P(X \neq 0 \text{ and } Y \neq 0) = p(1,1) + p(1,2) + p(2,1) + p(2,2) = .70.$

d. Compute the marginal pmf of X and of Y. Using $p_X(x)$, what is P(X <= 1)?

By summing row probabilities, $p_X(x) = .16, .34, .50$ for $x = 0, 1, 2$, By summing column probabilities, $p_Y(y) = .24, .38, .38$ for $y = 0, 1, 2$. $P(X \le 1) = p_X(0) + p_X(1) = .50.$

e. e. Are X and Y independent rv's?

$p(0,0) = .10$, but $p_X(0) \cdot p_Y(0) = (.16)(.24) = .0384 \neq .10$, so X and Y are not independent.

2. Two components of a computer have the following joint pdf for their useful lifetimes X and Y:

$$f(x, y) = \begin{cases} xe^{-x(1+y)} & x \geq 0 \text{ and } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

   a. What is the probability that the lifetime X of the first component exceeds 3?

$$P(X > 3) = \int_3^\infty \int_0^\infty xe^{-x(1+y)} dydx = \int_3^\infty e^{-x} dx = .050.$$

   b. What are the marginal pdf's of X and Y? Are the two lifetimes independent?

The marginal pdf of X is $f_X(x) = \int_0^\infty xe^{-x(1+y)} dy = e^{-x}$ for $x \geq 0$. The marginal pdf of Y is $f_Y(y) = \int_3^\infty xe^{-x(1+y)} dx = \dfrac{1}{(1+y)^2}$ for $y \geq 0$. It is now clear that $f(x,y)$ is not the product of the marginal pdfs, so the two rvs are not independent.

   c. What is the probability that the lifetime of at least one component exceeds 3?

$$P(\text{at least one exceeds } 3) = P(X > 3 \text{ or } Y > 3) = 1 - P(X \leq 3 \text{ and } Y \leq 3)$$
$$= 1 - \int_0^3 \int_0^3 xe^{-x(1+y)} dydx = 1 - \int_0^3 \int_0^3 xe^{-x} e^{-xy} dy$$
$$= 1 - \int_0^3 e^{-x}(1 - e^{-3x}) dx = e^{-3} + .25 - .25e^{-12} = .300.$$

3. An instructor has given a short quiz consisting of two parts. For a randomly selected student, let X = the number of points earned on the first part and Y = the number of points earned on the second part. Suppose that the joint pmf of X and Y is given in the accompanying table.

$p(x, y)$		y = 0	5	10	15
x	0	.02	.06	.02	.10
	5	.04	.15	.20	.10
	10	.01	.15	.14	.01

   a. If the score recorded in the grade book is the total number of points earned on the two parts, what is the expected recorded score E(X + Y)?

$$E(X+Y) = \sum\sum (x+y)p(x,y) = (0+0)(.02) + (5+0)(.04) + \ldots + (10+15)(.01) = 14.10.$$

   b. If the maximum of the two scores is recorded, what is the expected recorded score?

For each coordinate, we need the maximum; e.g., max(0,0) = 0, while max(5,0) = 5 and max(5,10) = 10. Then calculate the sum: $E(\max(X,Y)) = \sum\sum \max(x,y)\cdot p(x,y) =$ max(0,0)(.02) + max(5,0)(.04) + … + max(10,15)(.01) = 0(.02) + 5(.04) + … + 15(.01) = 9.60.

c. Compute the covariance for X and Y

$E(X) = 5.55$, $E(Y) = 8.55$, $E(XY) = (0)(.02) + (0)(.06) + \ldots + (150)(.01) = 44.25$, so $\text{Cov}(X,Y) = 44.25 - (5.55)(8.55) = -3.20$.